Twilight

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Twilight

Definition : Twilight

The dim light that we seen before the sun rise for some time and the dim light that we seen after the sun set for some time is called twilight . The morning twilight is called dawn and evening twilight is called dusk.

Twilight

How twilight is cause ?

As the sun goes below the horizon darkness does not fall-in instantaneously. This is because even after the sun set its rays fall on the atmosphere above the earth and of the light thus received a considerable portion is reflected or scattered in various directions Therefore there is some diffused light lasting for some time. The intensity of this light gradually diminishes and finally gives way to complete darkness.

Note

The twilight lasts as long as the sun is with in 18° below horizon.

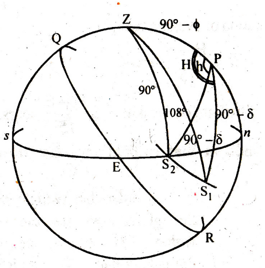
Theorem Blocks

Theorem (The duration of twilight)

The duration of twilight

Proof:

Let ϕ be the latitude of the place and δ be the declination of the sun on the date of observation.



Let S_1 be position of the sun 18° below the eastern horizon and S_2 be the position on the horizon.

Then S_1 marks of morning twilight and S_2 marks of evening twilight.

Let H and h be the hour angles of the sun at S_1 and S_2 respectively.

Now the duration of twilight is time taken by the sun to move from S_1 and S_2.

It is the time to describe the angle $\angle S_1PS_2$

ie the time required to describe hour angle $H - h$.

$\therefore$ The duration of morning twilight is $t = \frac{H-h}{15}$ hours.

Find H

From the spherical triangle $P S_1 Z$,

we have $PZ = 90° - \phi$, $PS_1 = 90° - \delta$, $ZS_1 = 108°$ and $\angle ZPS_1 = H$

Apply the cosine formula,

$$\cos ZS_1 = \cos PZ \cos PS_1 + \sin PZ \sin PS_1 \cos ZPS_1$$

$$\cos 108° = \cos(90° - \phi)\cos(90° - \delta) + \sin(90° - \phi)\sin(90° - \delta)\cos H.$$

$$-\sin 18° = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H.$$

$$\cos\phi \cos\delta \cos H = -(\sin 18° + \sin\phi \sin\delta)$$

$$\cos H = \frac{-(\sin 18° + \sin\phi \sin\delta)}{\cos\phi \cos\delta}$$

$$\therefore H = \cos^{-1}\left(\frac{-(\sin 18° + \sin\phi \sin\delta)}{\cos\phi \cos\delta}\right) \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (1)$$

Find h

In the spherical triangle PS_2Z

we have $ZP = 90° - \phi$, $PS_2 = 90° - \delta$, $ZS_2 = 90°$ and $\angle ZP S_2 = h$

Applying Cosine formula,

$$\cos ZS_2 = \cos ZP \cos PS_2 + \sin ZP \sin PS_2 \cos ZPS_2$$

$$\cos 90° = \cos(90° - \phi)\cos(90° - \delta) + \sin(90° - \phi)\sin(90° - \delta)\cos h.$$

$$0 = \sin\phi \sin\delta + \cos\phi \cos\delta \cos h.$$

$$\cos\phi \cos\delta \cos h = -\sin\phi \sin\delta$$

$$\cosh = -\frac{\sin\phi \;\sin\delta}{\cos\phi \;\cos\delta}$$

$$\cos h = -\tan\phi \tan\delta.$$

$$\therefore \quad h = \cos^{-1}[-\tan\phi \;\; \tan\delta] \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_ } (2)$$

The duration of morning twilight is $= \frac{H-h}{15}$ hours

where H and h are given by (1) and (2).

Similarly, the duration of evening twilight is also $= \frac{H-h}{15}$ hours

Therefore, The Total duration of the twilight on a day $= 2\left(\frac{H-h}{15}\right)$ hours

Note

Note

Duration of twilight is a function ϕ and δ. Therefore duration of twilight varies with the latitude of the place and declination of the sun. ie on the same date durations of twilight are different in different places, and for the same place durations of twilight are different on different dates.

Theorem Blocks

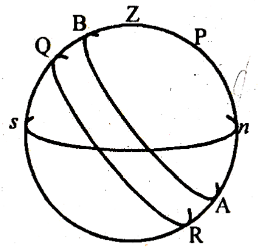
Theorem (The condition that twilight may last throughout night)

The condition that twilight may last throughout night

In order that twilight may last throughout the night. The distance of the sun below the horizon at its lower transit must be $\leq 18°$.

Let A be the position of the sun at the lower transit on any date and δ its declination on that date. Let ϕ be the latitude of the place.

$\therefore$ Condition for twilight to last throughout night,



$$nA \leq 18°$$

$$\text{ie} \quad PA - nA \leq 18°$$

$$PA - AR - nA \leq 18°$$

$$90° - \delta - \phi \leq 18°$$

$$90° - 18° \leq \phi + \delta$$

$$72° \leq \phi + \delta$$

$$\text{ie } \phi + \delta \geq 72°$$

$$\text{ie } \phi \geq 72° - \delta$$

Taking maximum value of $\delta = \omega$, then $\phi \geq 72° - \omega$.

Therefore twilight lasts throughout night only in places of latitude $\phi \geq 72° - \omega$ on the dates when the sun's declination is $\delta \geq 72° - \phi$.

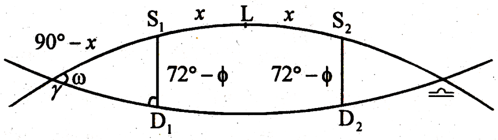
Theorem Blocks

Theorem (The number of consecutive nights having twilight throughout night)

The number of consecutive nights having twilight throughout night

For a places of latitude $\phi \geq 72° - \omega$, Twilight lasts throughout night when the sun's declination is $\delta \geq 72° - \phi$.

Therefore from the date the north declination of the sun is $72° - \phi$ to the date when it is again equal to $72° - \phi$ there will be twilight throughout night.



Let S_1 and S_2 be the positions of the sun when its north declination is $72° - \phi$.

The number of nights having twilight throughout night is the number of nights in the period which the sun takes to move from S_1 to S_2 .

Let L be the mid point of the ecliptic γ . L is also the mid point of S_1S_2.

Let D_1, D_2 be the feet of the declination circles through S_1 and S_2 to the equator. Let δ be the declination of the star and ϕ be the latitude of the place.

Let $S_1L = x$, so that $\gamma S_1 = 90° - S_1L = 90° - x$ and $S_1S_2 = 2x$.

From the spherical triangle $\gamma S_1 D_1$, we get

Apply Sine formula

$$\frac{\sin \gamma S_1}{\sin \ S_1 D_1 \gamma} = \frac{\sin \ S_1 D_1}{\sin S_1 \gamma D_1}$$

$$\frac{\sin (90° - x)}{\sin 90°} = \frac{\sin (72° - \phi)}{\sin \omega}$$

$$\sin (90° - x) = \sin (72° - \phi) \operatorname{cosec} \omega$$

$$\cos x = \sin(72° - \phi)\operatorname{cosec}\omega$$

$$\therefore\ x = \cos^{-1}[\sin(72° - \phi)\operatorname{cosec}\omega]$$

$$\therefore\ S_1S_2 = 2x = 2\cos^{-1}[\sin(72° - \phi)\operatorname{cosec}\omega]$$

Assuming that the motion of the sun along the ecliptic is uniform and the number of days in a year to be 365, the sun describes 360° in 365 days.

$$\therefore \text{ Time taken to describe } S_1S_2 = \frac{365}{360}\ S_1S_2 \text{ days}$$

$$= \frac{365}{360}\ 2\ x \text{ days} = = \frac{73}{36}\ x \text{ days}$$

$$= \frac{73}{36}\cos^{-1}[\sin(72° - \phi)\operatorname{cosec}\omega] \text{ days}$$

The number of nights having twilight throughout night is the integral part of $\frac{73}{36}\cos^{-1}[\sin(72° - \phi)\operatorname{cosec}\omega]$ days or the next integer.

Theorem Blocks

Theorem (The duration of twilight when it is shortest)

The duration of twilight when it is shortest

Let S_1 be the position of the sun at the beginning of twilight (ie 18° below the horizon) and S_2 its position on the horizon at the end of twilight.

Duration of twilight is the time taken by the sun to describe $\angle S_1PS_2$.

Rotate the celestial sphere westward, till S_1 is brought into coincidence with S_2.

Now the angle through which the celestial sphere is rotated is equal to Z_1.

As the celestial sphere is rotated the original position of zenith occupies a new position Z_1.

Let Z_1 be the zenith when S_1 is brought to S_2.

Now the angle described by the zenith at the pole is $\angle ZPZ_1 = \angle S_1PS_2$.

Therefore duration of twilight is Shortest, when $\angle ZPZ_1$ is minimum.

Clearly $\angle ZPZ_1$ is minimum when arc ZZ_1 is minimum.

In the spherical triangle $\angle ZZ_1S_1$

$$\text{arc } ZZ_1 + \text{arc } ZS_1 \geq \text{arc } Z_1S_1$$

$$\text{arc } ZZ_1 + 90^\circ \geq 108^\circ$$

$$\text{arc } ZZ_1 \geq 108^\circ - 90^\circ$$

$$\text{arc } ZZ_1 \geq 18^\circ$$

$\therefore$ Minimum value of arc $ZZ_1 = 18^\circ$.

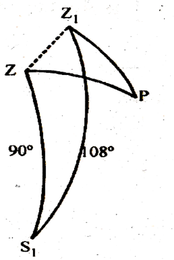
When arc $ZZ_1 = 18^\circ$, Z lies of Z_1S_1.

$\therefore$ For twilight to be shortest Z must be on Z_1S_1

Draw PD perpendicular to arc ZZ_1.

PD bisects $\angle ZPZ_1$ and arc ZZ_1.

$\therefore DZ = DZ_1 = 9^\circ$



Let ϕ be the latitude of the place and δ be the declination of sun on that day.

From spherical triangle PDZ

Here $PZ = (90° - \phi)$, $DZ = 9°$ and $\angle DPZ = 90°$

Applying Cosine formula,

$$\cos PZ = \cos PD \cos DZ + \sin PD \sin DZ \cos DPZ$$

$$\cos (90° - \phi) = \cos PD \cos 9° + \sin PD \sin 9° \cos 90°.$$

$$\sin \phi = \cos PD \cos 9°.$$

$$\therefore \quad \cos PD = \frac{\sin\phi}{\cos 9°} \text{ \_\_\_\_\_\_\_\_ } (1)$$

From spherical triangle PDS_1

Here $PS_1 = (90° - \delta)$, $DS_1 = 99°$ and $\angle PDS_1 = 90°$

Applying Cosine formula,

$$\cos PS_1 = \cos PD \cos DS_1 + \sin PD \sin DS_1 \cos PDS_1$$

$$\cos (90° - \delta) = \cos PD \cos 99° + \sin PD \sin 99° \cos 90°.$$

$$\sin \delta = \cos PD \cos 99°.$$

$$\therefore \quad \cos PD = \frac{\sin\delta}{\cos 99°} = -\frac{\sin\delta}{\sin 9°}$$

$$\therefore \quad \cos PD = -\frac{\sin\delta}{\sin 9°} \quad \_\_\_\_\_\_\_\_ \; (2)$$

From (1) and (2)

$$\frac{\sin\phi}{\cos 9°} = -\frac{\sin\delta}{\sin 9°}$$

$$\sin\delta = -\sin\phi \frac{\sin 9°}{\cos 9°}$$

$$\sin\delta = -\sin\phi \tan 9°$$

$$\therefore \; \delta = \sin^{-1}(-\sin\phi \; \tan 9°) \quad \_\_\_\_\_\_\_\_ \; (3)$$

$\therefore$Duration of twilight is shortest on the date when the sun's declination δ is given by $\delta = \sin^{-1}(- \sin\phi \; \tan 9°)$.

To find the shortest duration of twilight

Let $\angle ZPZ_1 = h$ say

so that shortest duration of twilight is $\frac{h}{15}$ hours.

PD bisects $\angle ZPZ_1$ $\quad\therefore \angle DPZ = \frac{h}{2}$.

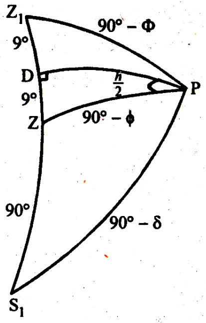
From spherical triangle PDZ,

Apply Sine formula,

$$\frac{\sin DPZ}{\sin DZ} = \frac{\sin PDZ}{\sin PZ}$$

$$\frac{\sin(h/2)}{\sin 9^\circ} = \frac{\sin 90^\circ}{\sin(90^\circ - \phi)}$$

$$\sin\frac{h}{2} = \sin 9^\circ \sec\phi$$



$$\frac{h}{2} = \sin^{-1}(\sin 9^\circ \quad \sec\phi)$$

$$h = 2\sin^{-1}(\sin 9^\circ \quad \sec\phi)$$

$\therefore$ Shortest duration of twilight $= \dfrac{\mathrm{h}}{15}$ hours.

$$= \frac{2}{15}\sin^{-1}(\sin 9^\circ \quad \sec\phi) \text{ hours.}$$

Hence the theorem.

Definition : Civil, Nautical and Astronomical twilights

The time when the centre of the sun is 6° below the horizon is called civil twilight. The name nautical twilight is applied to the time when the centre of the sun is 12° below the horizon. Astronomical twilight is the name given to the time when the sun is at a depth of 18° below the horizon. The Nautical Almanac gives the times of beginning of civil, nautical and astronomical twilights in morning and of their ending in the evening for different latitudes.

Problem

Problem : If the evening twilight ends when the sun is 18° below the horizon, show that at the equator the duration of evening twilight is given by

$$\frac{12}{\pi}\sin^{-1}(\sin 18^\circ \ \sec\delta\ )\,\text{hours.}$$

Soln:

The duration of evening twilight is $T = \dfrac{H-h}{15}$ hours.

$$\text{Where } \cos H = \frac{-(\sin 18^\circ + \sin\phi\ \sin\delta)}{\cos\phi\ \cos\delta} \text{ and } \cos h = -\tan\phi \tan\delta\,.$$

At equator $\phi = 0^\circ$.

$$\therefore \cos H = \frac{-(\sin 18^\circ + \sin 0^\circ\ \sin\delta)}{\cos 0^\circ\ \cos\delta} = \frac{-\sin 18^\circ}{\cos\delta} = -\sin 18^\circ\ \sec\delta$$

$$\cos H = -\sin 18^\circ\ \sec\delta \qquad \_\_\_\_\_\_\_\_ \ (1)$$

and $\cos h = -\tan 0° \tan \delta = 0.$

$$\cos h = 0. \quad \therefore \; h = 90° .$$

$$\therefore T = \frac{H - 90°}{15}$$

$$15\,T = H - 90°$$

$$15\,T + 90° = H.$$

$$\therefore \cos(15\,T + 90°) = \cos H.$$

$$-\sin 15T = -\sin 18° \sec\delta \quad \text{Using (1)}$$

$$\sin 15T = \sin 18° \sec\delta$$

$$15T = \sin^{-1}[\sin 18° \sec\delta]$$

$$T = \frac{1}{15}\sin^{-1}[\sin 18° \sec\delta] \text{ hours}$$

$$T = \frac{1x12}{15x12}\sin^{-1}[\sin 18° \sec\delta] \text{ hours}$$

$$= \frac{12}{180}\sin^{-1}[\sin 18° \sec\delta] \text{ hours}$$

At the equator, the duration of evening twilight = $T = \frac{12}{\pi}\sin^{-1}[\sin 18° \sec\delta]$ hours .

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Thank You